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Corrigendum

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In our paper we studied invariant Randers metrics on homogeneous Riemannian manifolds. There are some errors in the paper. Now we will correct these errors and also give some additional results.

The main point is that the Randers metrics constructed in theorem 2.2 need not necessarily be of Berwald type. For this an additional condition is needed. Now we give the correct version of this theorem

Theorem 2.2. *Let $\tilde{a}$ be an invariant Riemannian metric on G/H. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the inner product induced on $\mathfrak{g}$ by $\tilde{a}$. Then there exists a bijection between the set of all invariant Randers metrics on G/H with the underlying Riemannian metric $\tilde{a}$ and the set*

$$V_1 = \{X \in \mathfrak{m} \mid \mathrm{Ad}(h)X = X, \langle X, X\rangle < 1, \forall h \in H\}.$$

Moreover, for $X \in V_1$, the corresponding Randers metric is of Berwald type if and only if X satisfies

$$\langle [Y, X]_{\mathfrak{m}}, Z\rangle + \langle Y, [Z, X]_{\mathfrak{m}}\rangle + \langle [Z, Y]_{\mathfrak{m}}, X\rangle = 0, \quad \text{for any } Y, Z \in \mathfrak{m}, \qquad (*)$$

where $Y_{\mathfrak{m}}$ denotes the projection of Y to $\mathfrak{m}$ corresponding to (2.1). Furthermore, if G/H is not flat and $0 \neq X \in V_1$, then the corresponding Randers metric is neither Riemannian nor locally Minkowskian.

Proof. We only need to prove that $X \in V_1$ defines a Berwaldian metric if and only if X satisfies $(*)$. By theorem 11.5.1 of (Bao *et al* 2000 *An introduction to Riemann–Finsler Geometry* (Berlin: Springer)), the Randers metric is of Berwald type if and only if $\tilde{X}$ is parallel with respect to $\tilde{a}$, i.e. , if and only if

$$\tilde{a}(\nabla_{\tilde{Y}}\tilde{X}, \tilde{Z}\rangle = 0,$$

for any vector fields $\tilde{Y}$, $\tilde{Z}$ on G/H. Since $\tilde{a}$ is G-invariant, this holds if and only if

$$\langle \nabla_Y \tilde{X}, Z\rangle = 0, \qquad \forall\, Y, Z \in \mathfrak{m}.$$

By the formula for the Levi-Civita connection of an invariant Riemannian metric on a homogeneous manifold [1] (p 201), this is equivalent to the fact that X satisfies $(*)$. □

Since not all the invariant Randers metrics on homogeneous manifolds are of Berwald type, the formula for geodesics and flag curvature is not correct. We now give a correct version. Note that if the Randers metric is not of the Berwald type, then the connection is not the same as that of the underlying Riemannian metric and the flag curvature is generally very difficult to compute. Even if the Randers metric is of Berwald type, the connection is very complicated. So in the following, we assume that the Riemannian metric is naturally reductive with respect to the decomposition (2.1), i.e.,

$$\tilde{a}(X, [Z, Y]_{\mathfrak{m}}) + \tilde{a}([Z, X]_{\mathfrak{m}}, Y) = 0, \qquad \forall X, Y, Z \in \mathfrak{m}.$$

Theorem 3.1. *Let $(G/H, \tilde{a})$ be a naturally reductive homogeneous Riemannian manifold and $X \in V_1$ satisfying $(*)$. Let F be the corresponding Randers metric on G/H. Then we have the following:*

(i) The geodesics of F through the origin $o = eH$ are

$$\gamma_Y : \quad t \mapsto \exp tY \cdot o \quad (Y \in \mathfrak{m}).$$

(ii) Let Y be a nonzero vector in $\mathfrak{m}$ and P be a plane in $\mathfrak{m}$ containing Y. Then the flag curvature of the flag (P, Y) in $T_o(G/H)$ is given by

$$K(P, Y) = g_l\big(\tfrac{1}{4}[[U, l]_{\mathfrak{m}}, l]_{\mathfrak{m}} + [[U, l]_{\mathfrak{h}}, l], U\big),$$

where $l = \frac{Y}{\sqrt{g_Y(Y,Y)}}$, U is a vector in P such that U, l is an orthonormal basis of P with respect to g_l.

(iii) In particular, if $(G/H, \tilde{a})$ is a Riemannian symmetric manifold, then

$$K(P, Y) = g_l([[U, l], l], U).$$

Proof. The formula for geodesics and the curvature tensor of a naturally reductive homogeneous Riemannian manifold is given in [1] (p 202), from this the formula can be deduced directly. If the underlying Riemannian metric is symmetric, then it is naturally reductive and we have

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

From this (iii) follows. □

Remark. It is possible to give an explicit formula for $g_l(\cdot, \cdot)$. But the result is very complicated and we will not give it here.

References

[1] Kobayashi S and Nomizu K 1969 *of Differential Geometry* vol 2 (New York: Interscience)